Scale Dependent Intermittency and Conformal Invariance in Turbulence

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Abstract

We present a conformal theory for intermittent scalar fields. As an example, we consider the energy flux from large to small scales in the developed turbulent flow. The conformal correlation functions are found in the inertial range of scales. In the simplest case, the theory leads to the log-normal model. Parameters of the model are expressed via integrals of the conformal correlation functions. Non-Gaussian conformal correlation functions of high order are studied.

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1 Introduction

The present paper studies the scale and conformal symmetry of intermittent turbulent pulsations at very large Reynolds number. The scaling ideas were introduced into the turbulence theory by Kolmogorov [1]. Later those ideas were fruitfully explored in the second-order transition theory [2], in the quantum field theory and so on. The conformal symmetry is a powerful tool that strengthen the predictions of the scale symmetry. Its application to the turbulence theory is tangled by the scale dependent intermittency that destroys the simple scaling scheme.

Kolmogorov [1] defines the inertial range of scales for which he made two major suppositions. First, that the correlation functions of fluctuating fields are invariant to translations, rotations and scale transformations. Second, that the mean dissipation energy is the only dimensional parameter determining the statistics in that range. That theory was refined in 1962 [3] in order to take into account the effects of intermittency. The main result of the latter theory is the log-normal model that predict corrections to the simple scaling. In addition, a modification of that theory was proposed. The principal fields of the modified scaling theory was the ratios of the velocity differences. That fields were proposed to be statistically invariant according to scale transformations.

Scaling determines the correlation function up to unknown dimensionless universal functions. In order to obtain more information from symmetry groups, the conformal symmetry was proposed as the simplest extension of the scale one. The conformal group includes the special transformation that locally looks as the scale one. The conformal theories were applied to a wide set of physical problems [4], [5]. It is possible, in that theories, to determine the three point correlation function up to constants and to diminish the number of universal dimensionless arguments in the correlations of higher order. The conformal symmetry is especially informative in two dimensions where any analytic function of a complex variable induces some conformal transformation [6]. An application of the conformal symmetry to the fluid turbulence was considered in [7]. The conformal theory that based on the method of the paper [6] was studied in [8]. The present paper does not use the concepts of the latter method.

We consider the conformal theory for intermittent field in three dimensions. The starting point of the present theory is the Kolmogorov (1962) refined scaling for the dimensionless ratios of fields. We study application of

the conformal symmetry to correlation functions of energy dissipation. The result is logarithmical normal theory that relates correlations in space and scale, the expression for the constants of the log-normal model and expressions for correlations of higher order.

2 Simple and intermittent scaling

2.1 Simple scaling

The local structure of the developed turbulence has rotational and translational invariance. One considers the structure functions — the correlations of the velocity difference $\mathbf{w}(\mathbf{r}) = \mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})$. It is supposed that the structure functions are invariant to rotations and translations. In the simple scaling, the statistical regime is invariant to the scale transformations

$$\begin{aligned} \mathbf{x} &\to K\mathbf{x}, K > 0, \\ \mathbf{u}(\mathbf{x}) &\to K^{\Delta_u}\mathbf{u}(K\mathbf{x}), \end{aligned} \tag{1}$$

where Δ_u is a number called the scale dimension of velocity. In the Kolmogorov (1941) theory of incompressible fluid, $\Delta_u = -1/3$. The pair structure function is

$$\langle w_i(\mathbf{r}) w_j(\mathbf{r}) \rangle = A\varepsilon^{2/3} r^{2/3} \left(4\delta_{ij} - \frac{r_i r_j}{r^2} \right),$$

A is a constant.

2.2 Refined scaling for the dimensionless scalar fields

For simplicity, we consider the Kolmogorov modified theory for dimensionless ratios of the smoothed scalar fields. A reformulation of scaling for the vector fields see in [9]. To be definite, let us consider the density of the energy dissipation

$$\varepsilon(\mathbf{x}) = \frac{\nu}{2} \left(\frac{\partial u_i(\mathbf{x})}{\partial x_j} + \frac{\partial u_j(\mathbf{x})}{\partial x_i} \right)^2.$$

The dissipation smoothed over a sphere of radius l is

$$\varepsilon(\mathbf{x}, l) = \frac{3}{4\pi l^3} \int_{r \le l} \varepsilon(\mathbf{x}+\mathbf{r}) d^3 r.$$

This field is believed to model the energy flux from large scale to the small ones in the inertial range. Our approach is valid for any intermittent scale invariant scalar field.

The ratio of dissipation smoothed over two different scales l_1, l_2 is

$$\psi\left(\mathbf{x}, l_1, l_2\right) = \frac{\varepsilon(\mathbf{x}, l_1)}{\varepsilon(\mathbf{x}, l_2)}. \tag{2}$$

The dimensionless scalar $\psi\left(\mathbf{x}, l_1, l_2\right)$ is the field of the same kind as the Kolmogorov's ratios of the velocity differences. The present paper deals with inertial range where direct action of viscosity is negligible. According to the refined theory [3], the correlations of the field ψ have to be invariant to translations, rotations and to scale transformation. The scale transformation is

$$\psi\left(\mathbf{x}, l_1, l_2\right) \to \psi\left(K\mathbf{x}, Kl_1, Kl_2\right),$$

where $K > 0$ is any numeric multiplier. We suppose that the dimensionless fields have zeroth scaling dimension. Invariance to the scale transformation determines the pair correlations of ψ up to an universal scalar function Ψ of dimensionless variables

$$\left\langle \psi\left(\mathbf{x}, l_1, l_2\right) \psi\left(\mathbf{x} + \mathbf{r}, l_3, l_4\right) \right\rangle = \Psi\left(\frac{\mathbf{r}}{l_1}, \frac{l_2}{l_1}, \frac{l_3}{l_1}, \frac{l_4}{l_1}\right).$$

This formula has too many arguments. We define a simpler field that contains the same information as $\psi\left(\mathbf{x}, l_1, l_2\right)$.

From the definition of ψ, one has the identity

$$\psi\left(\mathbf{x}, l, l_2\right) = \psi\left(\mathbf{x}, l, l_1\right) \psi\left(\mathbf{x}, l_1, l_2\right).$$

Let us consider the limit $l_2 \to l_1 = l$. Expanding both sides of the identity in $\delta l = l_2 - l_1$, we have

$$\psi\left(\mathbf{x}, l, l_1\right) + \frac{\partial \psi\left(\mathbf{x}, l, l_1\right)}{\partial l_1} \delta l + \ldots = \psi\left(\mathbf{x}, l, l_1\right) \left[1 + \frac{\partial \psi\left(\mathbf{x}, l_1, l_2\right)}{\partial \ln l_2}\Big|_{l_2 = l_1} \frac{\delta l}{l_1} + \ldots\right],$$

and

$$\frac{\partial \psi\left(\mathbf{x}, l, l_1\right)}{\partial \ln l_1} = \varphi\left(\mathbf{x}, l_1\right) \psi\left(\mathbf{x}, l, l_1\right), \tag{3}$$

where $\varphi\left(\mathbf{x}, l_1\right) = \partial \psi\left(\mathbf{x}, l_1, l_2\right) / \partial \ln l_2 |_{l_2 = l_1}$. The dimesionless field φ is obtained from $\psi\left(\mathbf{x}, l_1, l_2\right)$ through infinitesimal displacement of l_2. Thus, its

correlation functions are scale invariant. If the statistics of φ were known, the correlations of ψ might be obtained from the Eq. (3). From (3), (2)

$$\varepsilon(\mathbf{x},l)=\varepsilon(\mathbf{x},L)\exp\left[-\int\limits_l^L\varphi(\mathbf{x},l_1)\frac{dl_1}{l_1}\right], \tag{4}$$

This equation determines the smoothed dissipation in terms of the scale invariant field φ.

2.3 Gaussian Self- Similar Model

Let L_0 be the largest scale in the inertial range. We suppose that L in the Eq. (4) is larger than L_0 so that $\varepsilon(\mathbf{x},L)=<\varepsilon>=\varepsilon_0$ has a constant value. In this subsection, $\varphi(\mathbf{x},l)$ is supposed to be a Gaussian field. Gaussian fields are determined by the correlation functions of first and second order. The mean value $\overline{\varphi}=\langle\varphi(\mathbf{x},l)\rangle$ is some constant in the inertial range as it follows from spatial homogeneity and scale invariance of φ.

$$\langle\varepsilon(\mathbf{x},l)\rangle=\varepsilon_0.$$

Let us take into account the conservation of the mean flow of energy along the scale axis. This leads to an additional relation that will be used later.

$$1=\left\langle\exp\left[-\int\limits_l^L\varphi(\mathbf{x},l_1)\frac{dl_1}{l_1}\right]\right\rangle. \tag{5}$$

For any Gaussian field $f(l)$ one has [10]

$$\begin{aligned}&\left\langle\exp\left[-i\int\limits_l^L\theta\left(l_1\right)f\left(l_1\right)dl_1\right]\right\rangle && (6)\\ =\ &\exp\left[-i\int\limits_l^L\theta\left(l_1\right)\langle f\left(l_1\right)\rangle\, dl_1-\frac{1}{2}\int\limits_l^L dl_1\int\limits_l^L dl_1\theta\left(l_1\right)\theta\left(l_2\right)\langle f\left(l_1\right)f\left(l_2\right)\rangle\right].\end{aligned}$$

If we choose here $f\left(l\right)=\varphi(\mathbf{x},l),\theta\left(l\right)=-i/l$, then from (5)

$$\exp\left[-\int\limits_l^L\langle\varphi(\mathbf{x},l_1)\rangle\frac{dl_1}{l_1}+\frac{1}{2}\int\limits_l^L\frac{dl_1}{l_1}\int\limits_l^L\frac{dl_2}{l_2}\langle\varphi(\mathbf{x},l_1)\varphi(\mathbf{x},l_2)\rangle\right]=1,$$

and

$$\int_{l}^{L} \frac{dl_1}{l_1} \int_{l}^{L} \frac{dl_2}{l_2} \langle \varphi(\mathbf{x}, l_1)\varphi(\mathbf{x}, l_2)\rangle = 2 \int_{l}^{L} \langle \varphi(\mathbf{x}, l_1)\rangle \frac{dl_1}{l_1}. \tag{7}$$

Formulae (4), (7) give

$$\left\langle \ln^2 \varepsilon(\mathbf{x}, l)\right\rangle = \ln^2 \varepsilon_0 + 2 \int_{l}^{L} \langle \varphi(\mathbf{x}, l_1)\rangle \frac{dl_1}{l_1}. \tag{8}$$

Let us divide the integration interval into (l, L_0) and (L_0, L). In the first (inertial) range $\langle \varphi(\mathbf{x}, l_1)\rangle$ is some constant $\overline{\varphi}$ as it was noticed above. The formula (8) is rewritten as

$$\left\langle \ln^2 \varepsilon(\mathbf{x}, l)\right\rangle = A + 2\overline{\varphi} \ln \frac{L_0}{l}, \tag{9}$$

where A origins from the contribution $A = \ln^2 \varepsilon_0 + 2 \int_{L_0}^{L} \langle \varphi(\mathbf{x}, l_1)\rangle \frac{dl_1}{l_1}$ that is not self-similar.

The similar formula had been derived using the log-normal model [11]

$$\left\langle \ln^2 \varepsilon(\mathbf{x}, l)\right\rangle = A(\mathbf{x}) + \mu \ln \frac{L}{l}.$$

Parameter μ is known to be equal $0.2 \div 0.4$. We see that it is universal and $\overline{\varphi} = \mu/2$.

In order to evaluate from (4) the spatial correlations of higher order, one needs the second moments of the Gaussian field φ. Translation, rotation and scale symmetries give for the pair correlations

$$\langle \varphi(\mathbf{x}, l_1)\varphi(\mathbf{x}+\mathbf{r}, l_2)\rangle = \Phi\left(\frac{r}{l_1}, \frac{l_2}{l_1}\right). \tag{10}$$

The correlation function is determined by the two dimensionless factors. In order to obtain an informative result, one needs to reduce the number of those factors. In the next section, that problem is solved with the help of the conformal invariance.

3 Simple and intermittent conformal invariance

Besides translations, rotations and the scale transformations, the conformal group in 3 dimensions includes the inversion about the unit circle

$$x'_i = x_i/x^2, \tag{11}$$

(for details see, for example, [4]). The conformal group is the simplest extension of the scale one.

To see this, let us consider the transformation of an infinitesimal displacement under the inversion. Let $\mathbf{y} = \mathbf{x} + \delta\mathbf{x}$, where $\delta\mathbf{x}$ is an infinitesimal displacement. The inversion transforms the vector $\delta\mathbf{x}$ according to

$$\delta x'_i = \frac{1}{x^2}\left(\delta_{ij} - 2\frac{x_i x_j}{x^2}\right)\delta x_j. \tag{12}$$

This transformation is the rotation by the orthogonal matrix

$$\Delta_{ij}(\vec{x}) = \delta_{ij} - 2\frac{x_i x_j}{x^2}. \tag{13}$$

and the dilatation in $1/x^2$ times. Therefore, the special conformal transformation (11) locally looks as a combination of rotation and dilatation.

The simple scaling is often complemented by the conformal invariance which is interpreted as the local scale invariance [5]. It has been proved mathematically that for a certain class of the Lagrangian field theories the conformal invariance follows from the scale one [4], [12].

3.1 Conformal theory for dimensionless fields

We defined the smoothed fields as integrals over the sphere of radius l. The scale transformation transforms l into Kl. While the conformal transformation, the dilatation factor K depends on the point. To find this dependance, let us consider the conformal transformation of the sphere

$$(\mathbf{x} - \mathbf{a})^2 = l^2.$$

After the special conformal transformation the center of the sphere and its radius become

$$\mathbf{a}_1 = \frac{\mathbf{a}}{a^2 - l^2}, \tag{14}$$

$$l_1 = \frac{l}{|a^2 - l^2|}. \tag{15}$$

We suppose that the correlation functions of φ are invariant to the transformations (14), (15). The scale l transforms like an additional imaginary coordinate. In this respect, our conformal transformation (14), (15) is similar to that in the relativistic field theories.

We suppose that the correlation functions of $\varphi(\mathbf{x}, l)$ are invariant to the above transformations. Translation, rotation and scaling symmetries has led to (10). The conformal invariance imposes the additional restriction. The function Φ may depend on the single parameter $(l_1^2 + l_2^2 - r^2)/l_1 l_2$. This parameter can be checked to be invariant to the conformal (14), (15) and to other above transformations. Therefore, the pair correlation have to be of the form

$$\langle \varphi(\mathbf{x}, l_1)\varphi(\mathbf{x}+\mathbf{r}, l_2)\rangle = \Phi\left(\frac{l_1^2 + l_2^2 - r^2}{l_1 l_2}\right). \tag{16}$$

3.2 Log-normal conformal theory for spatial correlations

In this subsection we suppose that $\varphi(\mathbf{x}, l)$ is not only invariant to conformal transformations but is Gaussian distributed also. Formula (4) gives

$$\langle \varepsilon(\mathbf{x}, l)\varepsilon(\mathbf{x}+\mathbf{r}, l)\rangle = \varepsilon_0^2 \left\langle \exp\left[-\int_l^L [\varphi(\mathbf{x}, l_1) + \varphi(\mathbf{x}+\mathbf{r}, l_1)]\frac{dl_1}{l_1}\right]\right\rangle .$$

With the help of (6), the mean of the exponent is written as the exponent of mean value of an expression. Using the Eq. (7), we have

$$\begin{aligned}
&\langle \varepsilon(\mathbf{x}, l)\varepsilon(\mathbf{x}+\mathbf{r}, l)\rangle \\
= \quad & \varepsilon_0^2 \exp\left\{-2\overline{\varphi}\ln\frac{L}{l} + \int_l^L \frac{dl_1}{l_1}\int_l^L \frac{dl_2}{l_2}\left[\begin{array}{c}\langle \varphi(\mathbf{x}, l_1)\varphi(\mathbf{x}, l_2)\rangle \\ +\langle \varphi(\mathbf{x}, l_1)\varphi(\mathbf{x}+\mathbf{r}, l_2)\rangle\end{array}\right]\right\} \\
= \quad & \varepsilon_0^2 \exp\left\{\int_l^L \frac{dl_1}{l_1}\int_l^L \frac{dl_2}{l_2}\langle \varphi(\mathbf{x}, l_1)\varphi(\mathbf{x}+\mathbf{r}, l_2)\rangle\right\}.
\end{aligned}$$

The last integrals is analyzed in polar coordinates λ, χ in l_1, l_2 space: $\lambda = \sqrt{l_1^2 + l_2^2}$, $\sin\chi = l_2/\sqrt{l_1^2 + l_2^2}$. The region of integration is divided in the 3 sub-regions 1) $l \le \lambda \le r$, 2) $r \le \lambda \le L_0$, 3) $L_0 \le \lambda \le L$.

The last region gives some non-universal contribution α_3. In that region the distance r$\ll\lambda$ and may be omitted. Thus, the dimensionless contribution α_3 is determined by the large scale structure and does not depend on r.

Sub-regions 1 and 2 belong to the inertial range. The scale and conformal invariance give in the polar coordinates

$$\langle\varepsilon(\mathbf{x},l)\varepsilon(\mathbf{x}+\mathbf{r},l)\rangle = \varepsilon_0^2\exp\left\{\alpha_3+2\int\limits_l^{L_0}\frac{d\lambda}{\lambda}\int\limits_0^{\frac{\pi}{2}}\frac{d\chi}{\sin 2\chi}\Phi\left[\frac{4}{\sin^2 2\chi}\left(\frac{\lambda^2-r^2}{\lambda^2}\right)^2\right]\right\}. \tag{17}$$

In the sub-region 2 the main logarithmical divergent term is extracted. In the remainder convergent contribution the upper limit is replaced by ∞. That approximation gives an error of the order of $O(r^2/L^2)$.

The result of the integration is

$$\langle\varepsilon(\mathbf{x},l)\varepsilon(\mathbf{x}+\mathbf{r},l)\rangle \approx C\varepsilon_0^2\left(\frac{L_0}{r}\right)^{\mu},$$

$$C = \exp\left[\sum_{i=1}^{3}\alpha_i(A)\right], \tag{18}$$

where $\alpha_i, i=1,2,3$ are determined by the integrals over the subregions 1,2,3.

$$\begin{aligned}\alpha_1(A) &= 2\int\limits_{l/r}^{1}\frac{ds}{s}\int\limits_0^{\frac{\pi}{2}}\frac{d\chi}{\sin 2\chi}\Phi\left[4\frac{s^2-1}{s^2\sin^2 2\chi}\right]\\ &\approx 2\int\limits_0^{1}\frac{ds}{s}\int\frac{d\chi}{\sin 2\chi}\Phi\left[4\frac{s^2-1}{s^2\sin^2 2\chi}\right] = const,\end{aligned}$$

$$\alpha_2(A) = 2\int\limits_1^{\infty}\frac{ds}{s}\int\limits_0^{\frac{\pi}{2}}\frac{d\chi}{\sin 2\chi}\left[\Phi\left(4\frac{s^2-1}{s^2\sin^2 2\chi}\right)-\Phi\left(\frac{-4}{\sin^2 2\chi}\right)\right] = const, \tag{19}$$

$$\alpha_3(A) = = 4\int\limits_{L_0}^{L}\frac{d\lambda}{\lambda}\int\limits_0^{\frac{\pi}{4}}\frac{d\chi}{\sin 2\chi}\left\langle\varphi(\mathbf{x},\lambda\cos\chi)\varphi(\mathbf{x}+\mathbf{r},\lambda\sin\chi)\right\rangle.$$

3.3 Spatial correlations of higher order

Formula (4) gives for the correlation function of nth order

$$\left\langle \prod_{i=1}^{n} \varepsilon(\mathbf{x}_i, l) \right\rangle = \varepsilon_0^n \left\langle \exp\left[-\int_l^L \sum_{i=1}^{n} \varphi(\mathbf{x}_i, l_1) \frac{dl_1}{l_1} \right] \right\rangle . \tag{20}$$

With the help of the Eq. (6), the mean of the exponent is written as the exponent of mean value of an expression. Using the Eq. (5), we have

$$\begin{aligned} & \left\langle \exp\left[-\int_l^L \sum_{i=1}^{n} \varphi(\mathbf{x}_i, l_1) \frac{dl_1}{l_1} \right] \right\rangle \\ = \quad & \exp\left\{ -n\overline{\varphi} \ln \frac{L}{l} + \frac{1}{2} \int_l^L \frac{dl_1}{l_1} \int_l^L \frac{dl_2}{l_2} \sum_{i=1}^{n} \sum_{j=1}^{n} \langle \varphi(\mathbf{x}_i, l_1) \varphi(\mathbf{x}_j, l_2) \rangle \right\} \\ = \quad & \frac{1}{2} \int_l^L \frac{dl_1}{l_1} \int_l^L \frac{dl_2}{l_2} \sum_{i \neq j}^{n} \Phi \left[\frac{l_1^2 + l_2^2 - r_{ij}^2}{l_1 l_2} \right], \end{aligned}$$

where $r_{ij}^2 = (\mathbf{x}_i - \mathbf{x}_j)^2$.

The integral is of the same kind as considered above. Similar straightforward algebra leads to

$$\left\langle \prod_{i=1}^{n} \varepsilon(\mathbf{x}_i, l) \right\rangle = C^n \varepsilon_0^n \prod_{i \neq j} \left(\frac{L_0}{r_{ij}} \right)^{\mu} .$$

3.4 Non-Gaussian conformal fields of higher order

Let us consider the correlation function of φ of the 3rd order

$$\Phi_3 \left(\mathbf{x}_1, l_1, \mathbf{x}_2, l_2, \mathbf{x}_3, l_3, \right) = \left\langle \varphi \left(\mathbf{x}_1, l_1 \right) \varphi \left(\mathbf{x}_2, l_2 \right) \varphi \left(\mathbf{x}_3, l_3 \right) \right\rangle .$$

Symmetry to translations, rotations and to the scale transformations lead to the form

$$\Phi_3 \left(\mathbf{x}_1, l_1, \mathbf{x}_2, l_2, \mathbf{x}_3, l_3, \right) = \Phi' \left(\frac{x_{12}}{l_1}, \frac{x_{13}}{l_1}, \frac{x_{23}}{l_1}, \frac{l_2}{l_1}, \frac{l_3}{l_1} \right),$$

where Φ' is some new universal function. There are three conformal invariants of the same kind as in the Eq. (16). The conformal symmetric function have

to be

$$\Phi_3\left(\mathbf{x}_1, l_1, \mathbf{x}_2, l_2, \mathbf{x}_3, l_3,\right) = \Phi''\left(\frac{l_1^2 + l_2^2 - x_{12}^2}{l_1 l_2}, \frac{l_1^2 + l_3^2 - x_{13}^2}{l_1 l_3}, \frac{l_3^2 + l_2^2 - x_{23}^2}{l_3 l_2}\right).$$

The generalization to the correlations of more high order is obvious. The correlation function have to depend on all independent conformal invariants.

4 Conclusions

We started from the modified Kolmogorov theory in terms of the ratios of smoothed fields. The scale symmetry determines the correlation functions of those fields as universal functions of dimensionless arguments. The differential equation (3) expresses the usual fields in terms on the scale invariant ones. Conformal symmetry diminished the number of dimensional arguments. For the Gaussian φ, the formulae of the log-normal model follows with definite expressions for its parameters in terms of the integrals of correlations of the conformal field φ. Experimental measuring of the correlations of φ seems to be necessary to check the proposed conformal symmetry. The generalization of the present theory to the vector fields is possible and will be considered in a separate paper.

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